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2002 J. Phys. A: Math. Gen. 35 5859

(http://iopscience.iop.org/0305-4470/35/28/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 5859–5874 PII: S0305-4470(02)31659-7

Transformation properties of perturbation expansions around elliptic fixed points

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Received 3 December 2001, in final form 2 May 2002
Published 5 July 2002
Online at stacks.iop.org/JPhysA/35/5859

Abstract

Transformation properties of perturbation expansions of vibrational quasi-periodic orbits upon the modular transformations of frequencies are studied, using simple but nontrivial examples: the Siegel complex quadratic map and special solutions of the real area-preserving quadratic map. It is shown that the transformation properties are similar to those in the previously studied case of the rotational invariant tori, except for some special features of the real map, related to the atypical nature of $1/3$ resonance in this case.

PACS numbers: 45.05.+x, 45.10.Hj
Mathematics Subject Classification: 40A05, 58F10, 70H99

1. Introduction

The major practical problem of Hamiltonian dynamics is to find efficient and accurate methods for computation of invariant tori, i.e., the invariant submanifolds of the phase space supporting regular quasi-periodic orbits of the system. The invariant tori are crucial in determining the domain of stability and properties of diffusion in phase space for Hamiltonian systems [1, 2]. On the other hand, calculations of tori are difficult in any realistic model even with the help of powerful computers. The crux of the difficulties is already present for the systems with only two degrees of freedom, and typical properties of these are conveniently studied using area-preserving maps [2, 3]. The standard approach is via canonical perturbation theory, which is designed to find a series expansion of a canonical transformation which conjugates the system on an interesting part of the phase space to a simpler integrable system. As is well known, due to the near-resonant orbits, or in other words, due to the famous small divisors, the formal series expansion might not correspond to a well-defined smooth canonical transformation. Depending on the formulation of the problem the perturbation series could be convergent or divergent. A general survey can be found, for example, in [1].

In order to make accurate estimates over a sufficiently long period of time relatively high orders of the perturbation expansion are needed. Furthermore, the perturbation expansion of a typical system on one part of its phase space, does not give any information about the perturbation expansion of the system on any other part of the phase space. Hard work computing the perturbation expansion to a sufficient order has to be done on each part of the phase space independently, and from the beginning.

On the other hand, the invariant tori with the same topological properties but on different distant parts of the phase space appear to be quite similar. There are two topologically different classes of invariant tori which also correspond to physically different motions. Rotational invariant tori (RIT) are those that are outside any of the resonant domains. In more technical terms they cannot be continuously transformed into a stable fixed point. The vibrational invariant tori (VIT) support the quasi-periodic small vibrations around an elliptic fixed point. Neighbourhoods of all RIT, on the one hand, and of the VIT on the other, look similar. This apparent global self-similarity of different parts of the phase space indicates that there should be a relatively simple relation between the perturbation expansions on different parts of the phase space.

The relation between the perturbation expansion of the RIT was studied in the method of modular smoothing [4–6]. It was shown that there are continuous and smooth relations between the corresponding coefficients of the perturbation expansions of the invariant tori with frequencies related by transformations of the modular group PSL$(2, Z)$. The method is summarized and discussed in the next section. Later, it was also shown that a similar relation exists between the actions of rotational periodic orbits [7].

In this paper we shall examine the transformation properties of the convergent perturbation expansions for the vibrational motion, that is around an elliptic fixed point. The simplest relevant examples are given by:

(a) the normal form of Siegel complex quadratic map

$$z_{j+1} = \exp(\mathrm{i}2\pi\nu_0)z_j + \tfrac{1}{2}z_j^2 \qquad z \in C \quad j \in Z \quad \mathrm{i} = \sqrt{-1} \tag{1}$$

(b) Lindstedt series of a special, complex, solution of the quadratic area-preserving map

$$\begin{aligned} q_{j+1} &= q_j \cos 2\pi\nu_0 - \left(p_j - q_j^2\right)\sin 2\pi\nu_0 \\ p_{j+1} &= p_j \sin 2\pi\nu_0 + \left(p_j - q_j^2\right)\cos 2\pi\nu_0 \qquad (q_j, p_j) \in R \times R \quad j \in Z. \end{aligned} \tag{2}$$

The perturbation theory for these maps will be recapitulated in section 3. The two perturbation expansions are convergent with the corresponding radius of convergence $\rho(\nu_0)$, which is a complicated fractal function of the frequency ν_0 of the linear rotation around the elliptic fixed point at the origin. We shall see, in section 4, that there are continuous but non-differentiable functions which describe the transformation properties of $\rho(\nu_0)$ under the action of the modular group on ν_0. Then, we shall examine the relations between the coefficients of the corresponding perturbation series.

2. Modular smoothing and the rotational invariant tori

In this section we summarize the basic ideas and the results of the method of modular smoothing, as applied to the RIT of the standard map, semi-standard map and the analogous continuous time systems.

First, we recapitulate the definitions of the systems that have been studied and briefly review the Lindstedt series for the RIT of these systems. All this is quite well known and has been reviewed many times [3, 8].

The standard map (SM) is an area-preserving map of the cylinder given by

$$p_{j+1} = p_j + \frac{k}{2\pi}\sin 2\pi q_j \qquad q_{j+1} = q_j + p_{j+1} \tag{3}$$

where $p_j, q_j \in R \times (R/2\pi Z) \equiv R \times T^1$.

We shall also use the so-called semi-standard map (SSM), which is obtained from (3) by replacing $\sin(2\pi q)$ in the first part of equation (3) by $\exp(\mathrm{i}2\pi q)$:

$$p_{j+1} = p_j + \frac{k}{2\pi}\exp \mathrm{i}2\pi q_j \qquad q_{j+1} = q_j + p_{j+1}. \tag{4}$$

The RIT $T_{\nu,k}$ for SM or SSM with the frequency ν, established by the KAM theorem, are given by a function $u(\theta; k, \nu)$, such that a simple change of the coordinate $\theta \to q$ on T^1, given by

$$q = \theta + u(\theta; k, \nu) \tag{5}$$

gives a quasi-periodic solution of (3) or (4) with the frequency ν.

$$q_j = 2\pi\nu j + u(2\pi\nu j; k, \nu) \qquad p_j = q_j - q_{j-1}. \tag{6}$$

For the SSM $u(\theta; k, \nu) \equiv \mathrm{i}\kappa(\theta; k, \nu)$ is a complex function. The function $u(\theta; k, \nu)$ for fixed ν and k conjugates the dynamics on the RIT with the frequency ν to a linear rotation with the same frequency.

If ν is a fixed irrational, satisfying a Diophantine condition, and if k is sufficiently small $k < K(\nu)$, the function u is analytic in θ (and in $k \in [0, K(\nu))$). However, it is a very complicated fractal function of ν. For map (3) or (4) and any Diophantine ν there is a corresponding unique $K(\nu)$ called the breakdown threshold or the critical value of k, for which u becomes a non-differentiable function of θ, and for $k > K(\nu)$ the RIT is replaced by a fractal discontinuous structure called cantorus [9–11]. The breakdown threshold as a function of ν is called the critical function [12, 13]. By definition $K(m/n) = 0$ for all rational frequencies m/n (in all that follows m and n are always relatively prime). Since u, as a function of ν, has an everywhere dense, zero measure set of singularities at frequencies which are rational m/n, or irrationals very well approximated by rationals, the function $K(\nu)$ is a very complicated fractal function. We shall say more about $K(\nu)$, but we first discuss the most straightforward way to estimate $K(\nu)$ using the perturbation theory.

The equation that determines u is easily obtained by writing (3) or (4) in the Lagrangian form, that is as a second difference equation. The functional equation satisfied by u is

$$u(\theta + 2\pi\nu; k, \nu) - 2u(\theta; k, \nu) + u(\theta - 2\pi\nu; k, \nu) = k\sin(\theta + u(\theta; k, \nu)) \quad (\text{for } SM) \tag{7}$$

or

$$u(\theta + 2\pi\nu; k, \nu) - 2u(\theta; k, \nu) + u(\theta - 2\pi\nu; k, \nu) = (\mathrm{i}k\exp(\theta + u(\theta; k, \nu)) \quad (\text{for } SSM) \tag{8}$$

Equation (7) or (8) has a formal solution as a power series in k of the form

$$u(\theta; k, \nu) = \sum_{n\geqslant 1} u_n(\theta)k^n = \sum_{n\geqslant 1}\sum_{m\in Z} b_{n,m}(\nu)k^n \exp(\mathrm{i}m2\pi\theta) \tag{9}$$

where b-coefficients are calculated recursively. The Taylor–Fourier expansion (9) is called the Lindstedt series of the invariant tori $T_{\nu;k}$. In the case of SM $|m| \leqslant n$, and in the case of the SSM the double expansion (9) collapses into a single power expansion in the new variable $r = k\exp \mathrm{i}\theta$, i.e. in the case of SSM one has the solution $\kappa = -\mathrm{i}u$ in the form

$$\kappa(\theta; k, \nu) = \sum_{n\geqslant 1} b_n(\nu)r^n. \tag{10}$$

The recursion relations for b-coefficients for SM are

$$\text{for } m \neq 0 \qquad b_{n,m} = \frac{1}{D(n;\nu)} \sum_{l\geqslant 0} \frac{1}{l!} \sum_{m_0+\cdots+m_l=m} \sum_{n_1+\cdots+n_l=n-1} \frac{1}{2}(-\mathrm{i}m_0)(\mathrm{i}m_0)^l \prod_{j=1}^{l} b_{n_j,m_j}$$
$$D(n;\nu) = 4\sin^2(\pi\nu n) \qquad m_0 = \pm 1, \tag{11}$$

and for $m = 0$ and for all $n \geqslant 1$, $b_{k,0} = 0$. The term in (11) corresponding to the index $l = 0$ is by definition such that $n = 1$ and $m = m_0$ and, then $b_{1,m_0} = -\mathrm{i}m_0/D(1,\nu)$.

For SSM the recursion relations are simpler:

$$b_1 = 1 \qquad b_n = \frac{c_{n-1}}{D(n;\nu)} \qquad c_n = \frac{1}{n}\sum_{m=1}^{n} m b_m c_{n-m} \qquad D(n;\nu) = 4\sin^2(n\pi\nu). \tag{12}$$

Note the same form of the divisors $D(n;\nu)$ for both maps.

The radius of convergence $\rho(\nu)$ of the series (9) can be calculated as

$$\rho(\nu) = \inf_{\theta\in T^1} \limsup_{n\to\infty} |b_n(\theta;\nu)|^{-1/n} = \limsup_{n\to\infty} \max_{|m|\leqslant n} |b_{n,m}(\nu)|^{-1/n} \tag{13}$$

for SM, and as

$$\rho(\nu) = \lim_{n\to\infty} (b_n(\nu))^{-1/n} \tag{14}$$

for SSM, since all b-coefficients are positive for SSM.

Conjugation functions for the rotational invariant tori and their Lindstedt series, with the corresponding radius of convergence, are analogously defined, and have analogous properties for continuous time systems, like the following two standard examples [14–16]:

$$H = p_1^2/2 + 2\pi p_2 + \frac{k}{2\pi}(\sin(2\pi q) + \sin 2\pi(q_1 - q_2)) \tag{15}$$

and

$$H = p_1^2/2 + 2\pi p_2 + \frac{k}{2\pi}(\exp(\mathrm{i}2\pi q) + \exp \mathrm{i}2\pi(q_1 - q_2)) \tag{16}$$

which we call the two-wave model (TWM) and the complex two-wave model (CTWM).

The exact relation between $\rho(\nu)$ and $K(\nu)$ is not straightforward. The relation can be analysed by studying the domain of analyticity of $u(\theta; k, \nu)$ in the complex θ as $k \to \rho(\nu)$ [12], or by studying the domain of analyticity in the complex k. This has been investigated for SSM and SM and a few other area-preserving maps on the cylinder [17, 18]. The boundary of analyticity was always found to be a natural boundary but it might be of a non-circular shape, so in general $\rho(\nu) \leqslant K(\nu)$, the latter being the (real) number where the boundary of analyticity intersects the real axes. However, for SSM one has that $\rho(\nu) = K(\nu)$ [19], and in the case of SM, numerical evidence [17, 18] suggests that $\rho(\nu) = K(\nu)$ if ν is the golden mean $\gamma = (\sqrt{5}-1)/2$ or any other, so-called noble irrational, i.e. a number with the continued fraction expansion of the form

$$\nu = \frac{1}{a_1 + \cdots + \frac{1}{a_n+\gamma}} \equiv \{0, a_1, \ldots, a_n, 1, 1, \ldots\}. \tag{17}$$

However, it has also been shown that there are irrational numbers ν (not of the form (17)), such that the ratio $\rho(\nu)/K(\nu)$ for SM can be an arbitrary small positive number.

Nevertheless, the qualitative properties of $\rho(\nu)$ and $K(\nu)$ are similar. They are both zero at rationals and some irrationals, and different from zero at all Diophantine irrationals, in particular, at the noble irrationals. The results of the method of modular smoothing are obtained using always the noble numbers and are formulated in terms of either $\rho(\nu)$ or $K(\nu)$. In our analysis we shall again use only noble numbers.

In order to calculate $\rho(\nu)$ (or the RIT itself) with sufficient accuracy, one needs to work through the recurrence relations up to a quite high order. Also the b-coefficients of T_ν for different ν are not related in an obvious way. On the other hand, the global similarity of neighbourhoods of distant RIT with different noble frequencies, which is also indicated by the self-similarity of $\rho(\nu)$ around different distant nobles, indicates that there should be a relation between the perturbation expansions (9) for different noble RIT. In particular, $\rho(\gamma)$ should be simply related to $\rho(\nu)$ for all other noble ν. All noble numbers can be obtained from the golden mean by applications of elements of the modular group PSL(2, Z), i.e. by successive applications of its two generators

$$R_1\nu = \nu + 1 \qquad I\nu = -1/\nu. \tag{18}$$

In fact, the set of nobles is also invariant under the action of PSL(2, Z). This was the motivation in [4] to study the transformation properties of $\rho(\nu)$ under the action of PSL(2, Z). The properties of $\rho(\nu)$ under the action of R_1 and $\nu \to -\nu$ are trivial, so one has to explore the relation between $\rho(\nu)$ and $\rho(1/\nu)$. The essential observation for the idea of modular smoothing, as applied to $\rho(\nu)$ (or $K(\nu)$), is that the singularities at different rationals, and in particular at m/n and n/m, are of the same form. In fact in [4] it was assumed that the function $L_0(\nu) = \ln\rho(\nu)$ (for SSM) has the following form near singularities:

$$L_0(\nu) \approx \frac{A_0}{n} Q(\nu - m/n) + A_1(m/n) \qquad \text{as} \quad \nu \to m/n \tag{19}$$

and

$$L_0(\nu) \approx \frac{A_0}{m} Q(1/\nu - n/m) + A_2(n/m) \qquad \text{as} \quad \nu \to n/m \tag{20}$$

where $A_1(m/n)$, $A_2(m/n)$ and A_0 are bounded, A_0 is a constant, and the function $Q(\nu)$ is not specified, but it is assumed that it has singularities weaker than a pole. The same form of the singularities of $\rho(\nu)$ or $K(\nu)$ for SM and the continuous time systems (15) and (16) was also implied by applications of the modular smoothing method to these systems [5, 6]. This form of the singularities of $\rho(\nu)$ has been obtained recently, by studying $\rho(m/n + \mu)$ where μ is complex and $\mu \to 0$ over paths which are not tangent to the real axes [20–22].

If (19) and (20) are correct then the function

$$L_1(\nu) = L_0(\nu) - \nu L_0(1/\nu) \tag{21}$$

is bounded except at 0 and infinity. Numerical evidence presented in [4] for the SSM (and in [6] for CTWM) shows that $L_1(\nu)$ is actually continuous but with singularities in the first derivative at every rational ν. The function $L_1(\nu)$ for SM and TWM, defined directly in terms of $K(\nu)$ (rather than $\rho(\nu)$ as for SSM) is also bounded except at 0 and infinity and continuous with singularities in the first derivative [5]. Furthermore, it was shown (for SM and TWM in [5] and for CTWM in [6]) that the value of L_1 at a rational m/n is given by the ratio of the b-coefficients of the suitable perturbation expansions of orders n and m. For example, in the case of SSM the formula

$$\mathcal{L}_1(m/n) = \lim_{\nu\to m/n} \frac{1}{n} \ln\left[\frac{m}{n}\frac{b_m(\nu^{-1})}{b_n(\nu)}\right] \tag{22}$$

gives a function which appears to be continuous over rationals, and is equal to $L_1(m/n)$. The functions $\rho(\nu)$ and $L_1(\nu)$ for SSM are illustrated in figures 1(a) and (b).

Conclusions about the properties of $L_1(\nu)$ are based on numerical evidence and analysis of the perturbation expansions. In [23] the properties of L_1 are related to a (conjectured) relation between the critical function and the exponent of the so-called Bruno function. This last relation has been proved recently in [22].

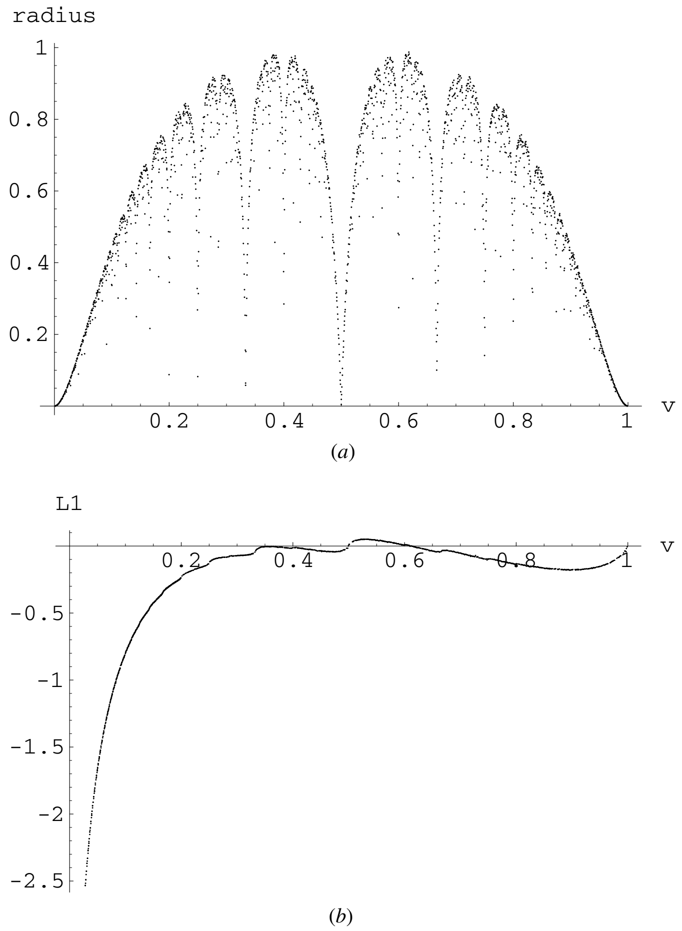

Figure 1. (*a*) The radius of convergence $\rho(\nu)$ and (*b*) the function $L_1(\nu)$ for the semi-standard map.

Formula (21) gives transformation properties of the fractal functions $\rho(\nu)$ or $K(\nu)$ in terms of the continuous function, which can be approximated by some smooth approximation, and used for efficient and accurate computation of the critical function for any noble ν, given one value of $K(\gamma)$ [4, 5]. In the same spirit, relation (22) can be used for approximate calculations of the RIT [6]. In this case formula (22) gives a reasonable approximation for RIT close to the resonances, for which the corresponding b coefficients dominate the other non-resonant contributions. For RIT far from the resonances some form of higher order smoothing is needed [6].

The idea of modular smoothing was also used to study other fractal functions related to the rotational motion. In [7] it was applied to find continuous relations between the critical values $k_c(m/n)$, for which elliptic rotational periodic orbits of SM with frequencies (m/n) bifurcate into hyperbolic (with reflection) rotational periodic orbits with the same frequency. It was also used to study transformation properties of the action $A(k; m/n)$ of periodic orbits, which is a smooth function of k but a fractal function of m/n [24].

The message that we can extract from studying the RIT in this section is that the perturbation expansions characterized by the fixed frequency (one might call them isochronous expansions) have simple transformation properties with respect to the action of the modular group on ν, which lead to useful relations between the coefficients of the expansions for different ν.

3. Examples of fixed frequency expansions for vibrational motion

Small vibrations around an equilibrium of a system of oscillators given by the Hamilton function

$$H = \nu_0 p + k f_1(p, q) + k^2 f_2(p, q) + \cdots \tag{23}$$

are restricted to the VIT. If the system is linear the frequencies of motion on different VIT are equal, and given by the masses and Jung's constants of the oscillators. For nonlinear systems the frequencies are, in general, non-constant functions of action, i.e., of the tori. The standard way to approach the small nonlinear terms in (23) is via transformation into the Birkhoff normal form [1]. If the linear frequencies ν satisfy some Diophantine condition, then one can push the angle dependence to arbitrary high finite order in k. For example, one can construct a smooth canonical transformation such that the new Hamilton function in the new coordinates (which we denote again by (p, q)) is

$$H^{(n)} = \nu_0 p + k H_1(p) + \cdots + k^n H_n(p) + k^{n+1} f_{n+1}^n(p, q) + \cdots. \tag{24}$$

However, it is well known that the series expansion of this canonical transformation is generally divergent when the order $n \to \infty$. On the other hand, if the nonlinear terms are such that $h_1(p) = h_2(p) = \cdots = 0$ to all orders, then the Birkhoff canonical transformation is convergent and the system can be linearized by a smooth canonical transformation [25]. The system is then integrable (according to Birkhoff), and the frequencies on all VIT are equal to the linear frequency ν_0. In this case the construction of the Birkhoff transformation is an example of an expansion with fixed frequency. However, since the system is then integrable for any Diophantine linear frequency ν_0, the transformation of VIT T_{ν_0} upon modular transformations of ν_0 (i.e. upon changing the masses and constants of the oscillators) is trivial. We need examples of fixed frequency perturbation expansions for non-integrable dynamical systems around an elliptic fixed point.

As the first simplest but nontrivial example of a dynamical system with an elliptic fixed point at the origin and an isochronous perturbation expansion we consider the Siegel complex quadratic map (1): $z' = \exp(\mathrm{i}2\pi\nu)z + z^2/2$, where we shall skip the subscript 0 on the linear frequency. If the frequency ν satisfies a Diophantine condition then the nonlinear map (1) is analytically conjugate to its linear part on a finite domain around the fixed point. More precisely, there is an analytic function $\Phi(\zeta; \nu) : \zeta \to z$ such that

$$f \circ \Phi(\zeta; \nu) = \Phi(\exp(\mathrm{i}2\pi\nu)\zeta; \nu) \tag{25}$$

The domain of analyticity of Φ is a disc $D_{\rho_S(\nu)} = \{\zeta \in C : |\zeta| < \rho_S(\nu)\}$, on which Φ is given by the Taylor expansion

$$\Phi(\zeta; \nu) = \zeta + \sum_{k=2}^{\infty} b_n(\nu)\zeta^n \equiv \Phi(\theta; \nu, k) = \sum_{k=1}^{\infty} b_n(\nu)k^n \exp(\mathrm{i}\theta) \qquad \zeta \equiv k \exp(\mathrm{i}\theta) \tag{26}$$

with the radius of convergence:

$$\rho_S(\nu) = \lim_{n\to\infty} \sup |b_n(\nu)|^{-1/n} \tag{27}$$

which is called the Siegel radius.

The b-coefficients satisfy a simple recursion relation:

$$b_n(\nu) = D(n,\nu)^{-1} \sum_{m\geqslant 2}^{n} b_m(\nu) b_{n-m}(\nu) \qquad b_1(\nu) = 1$$
$$D(n,\nu) = (\exp(\mathrm{i}n2\pi\nu) - \exp(\mathrm{i}2\pi\nu)) \tag{28}$$

and the parameter $k < \rho_S(\nu)$ is arbitrary.

For any fixed $k < \rho_S(\nu)$ expansion (26) represents the Lindstedt series of a vibrational invariant (complex) analytic circle with a frequency equal to the linear frequency ν. That is, the curves $T_{\nu;k} = \{z : z = \Phi(k\exp(\mathrm{i}2\pi\theta);\nu), \text{fixed}\, k < \rho_S(\nu), \theta \in S^1\}$, that are images, by $\Phi(\zeta;\nu)$, of circles with any radius less than $\rho_S(\nu)$, are vibrational invariant analytic curves filled by quasi-periodic orbits with constant frequency ν of map (1). The circles $\{T_{\nu,k}, k < \rho_S(\nu)\}$ foliate the domain $U = \Phi(D_{\rho_S(\nu)})$, called the Siegel domain, on which map (1) is conjugate to the rotation with fixed frequency ν.

In order to compute $\rho_S(\nu)$ one can apply the definition (27), but the convergence is rather slow. On the other hand, it has been shown that $\rho_S(\nu)$ could be computed much more efficiently using an averaging procedure given by the following formula [26]:

$$\ln \rho_S(\nu) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \ln\left(\left|f^{(j)}(z_0)\right|\right) \tag{29}$$

where $z_0 = -\exp(\mathrm{i}2\pi\nu)$. Formula (29) is a consequence of the facts that the quasi-periodic orbit is ergodic on the invariant circle and that a critical point z_0 lies on the boundary of the Siegel domain.

The convergence radius $\rho_S(\nu)$ as a function of ν, illustrated in figure 2(*a*), is similar to the analogous functions for SSM and SM. Actually the problem of linearization of the Siegel quadratic map is the simplest problem with small divisors, and has been studied thoroughly (a review and references can be found in [27]). For our purposes it is important to note that the sets on which $\rho_S(\nu)$ is zero or different from zero are both invariant under PLS(2, Z). Furthermore, one knows [28, 27] that there is a function $B(\nu)$, defined by the continued fraction expansion of ν and independent of the dynamics, such that $(\rho_S(\nu))^{-1}\exp(-B(\nu))$ is a bounded function of ν. This is also true, with $B(\nu)$ replaced by $2B(\nu)$, for SSM [19], and has been conjectured for SM in [23] and recently proved in [22]. Thus, one could study singularities of $\rho_S(\nu)$ by studying those of $B(\nu)$. Furthermore, $B(\nu) - \nu B(\nu) = \ln 2$, for $\nu < 1/2$, and this relation has been used to provide an alternative explanation for the properties of the $L_1(\nu)$ for SSM [23]. However, we shall investigate the transformation properties of $\rho_S(\nu)$, with no reference to the relation between $\rho_S(\nu)$ and the function $B(\nu)$.

The second example that we analyse is the real area-preserving quadratic map (2) with an irrational Diophantine linear frequency ν_0. It is convenient to write the map in the following Hamiltonian form:

$$p_{i+1} = p_i + C(\nu_0)q_i + q_i^2 \qquad q_{i+1} = p_i + (1 + C(\nu_0))q_i + q_i^2 \tag{30}$$

or in the equivalent Lagrangian form

$$q_{i+1} - 2q_i + q_{i-1} = C(\nu_0)q_i + q_i^2 \tag{31}$$

where $C(\nu_0)$ is fixed by the frequency of the linear rotation ν_0 in (2) as $C(\nu_0) = 4\sin^2\pi\nu_0$.

Invariant tori with quasi-periodic orbits of (2) with fixed frequency ν are given as solutions of the following functional equation:

$$q(\theta+\nu;\nu) - 2q(\theta;\nu) + q(\theta-\nu;\nu) = C(\nu_0)q(\theta;\nu) + q(\theta;\nu)^2. \tag{32}$$

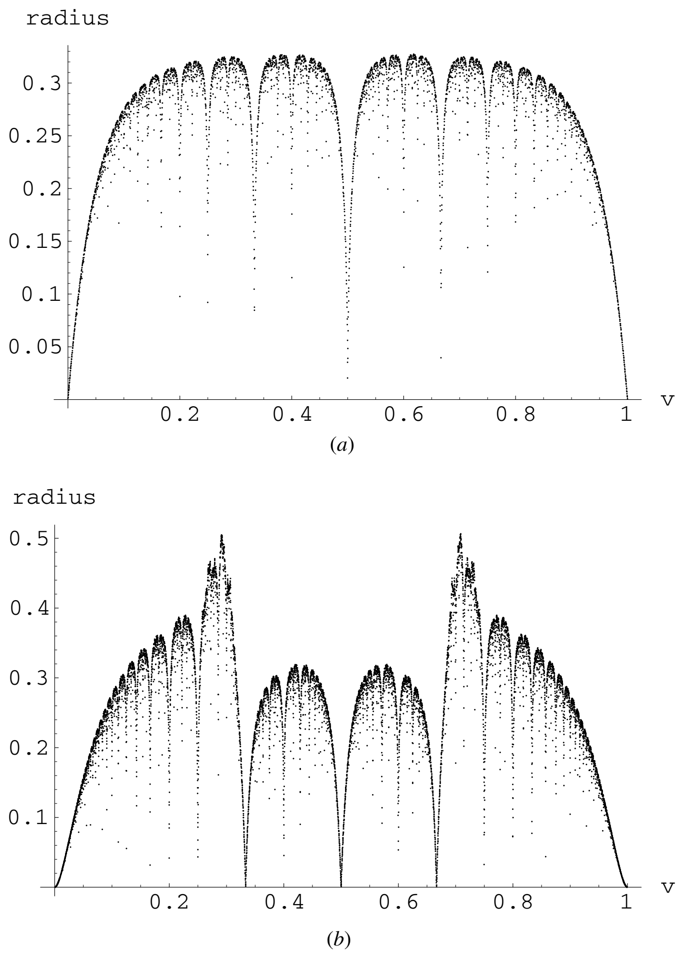

Figure 2. The radii of convergence for the (*a*) complex quadratic maps ($\rho_S(\nu)$) and (*b*) special solutions of the real area-preserving quadratic maps ($\rho_H(\nu)$).

An application of the KAM theorem ensures the existence of the solutions of these equations for a sufficiently small range of frequencies near ν_0. However, in our example we shall use a very special solution of (32) (first studied in [29]), namely the one with the frequency ν equal to the linear frequency $\nu = \nu_0$, and we shall later study the transformation properties of the expansion of this solution as ν_0 (and always $\nu = \nu_0$) is changed by modular transformations.

So, let us look for a solution of (32), $q(\theta; \nu)$, analytic in the complex θ which gives a quasi-periodic solution of equation (31) with the frequency equal to the linear frequency of the map: $\nu = \nu_0$ [29]. We attempt to find such solution by substituting a Fourier series:

$$q(\theta; \nu) = \sum_{n \in Z} \alpha_n(\nu) \exp(\mathrm{i}n\theta) \tag{33}$$

into (32), where $\nu = \nu_0$. The last requirement is equivalent to $D_1 \equiv 4\sin^2(\pi\nu) = -C(\nu_0)$. From now on $\nu = \nu_0$ and we skip the subscript 0. Then the coefficients $\alpha_n(\nu)$ satisfy the following recursion:

$$\text{for } n > 1 \qquad \alpha_n(\nu) = (D_1 - D_n)^{-1} \sum_m^{n-1} \alpha_m(\nu)\alpha_{n-m}(\nu) \qquad D_n = 4\sin^2(n\pi\nu) \tag{34}$$

and

$$\text{for } n < 0 \qquad \alpha_n(\nu) = 0 \tag{35}$$

where $\alpha_1(\nu) \equiv k$ is a free parameter.

The special solution $q(\theta;\, \nu)$ given by (33)–(35) is complex and analytic in the half plane. It has the frequency ν equal to the linear frequency ν_0, and for real θ it exists only at the stable fixed point of the map at the origin. This is similar to SSM whose invariant tori are also complex functions of the complex θ, analytic on the half plane. On the other hand, map (2) is real (while SSM is complex and for example different from SM) but the solution that we analyse is nontrivial only for complex θ. Complex solutions of the complexified quadratic area-preserving map (2), were analysed in [30], and have similar, but also different, properties from the solutions considered here (and in [29]).

In order to write expansion (33) in the form of the Lindstedt series for $q(\theta;\, \nu)$ one can introduce the coefficients $b_n(\nu)$ as equal to $\alpha_n(\nu)$ when $\alpha_1(\nu) \equiv k = 1$. Then

$$q(\theta;\, \nu, k) = \sum_{n\geqslant 1} b_n(\nu) k^n \exp \mathrm{i} n\theta. \tag{36}$$

The series (36) is convergent within the circle of convergence with the radius

$$\rho_H(\nu) = \lim_{n\to\infty} \sup |b_n(\nu)|^{-1/n} \tag{37}$$

and for

$$\operatorname{Im}\theta > \ln(k/\rho_H(\nu)). \tag{38}$$

The natural boundary (in complex θ-plane) of convergence of (36) coincides with the real θ-axis when the parameter $k = \rho_H(\nu)$, and for this value of k the analytic solution $q(\theta;\, \nu, \rho_H(\nu))$ vanishes for real θ. Thus the parameter k plays the same role as the parameter k of SSM, and its critical value can be estimated by (37).

The function $\rho_H(\nu)$ is illustrated in figure 2(*b*). As all other ρ functions considered here, it is zero at the rationals and nonzero at most of the irrationals. However, there are some obvious differences. For example, and this is not that important, the absolute maximum of $\rho_H(\nu)$ is not at the golden mean. The most important difference, however, is the well-known fact (for example [31] and the references therein) that the linear frequency $\nu = 1/3$ is obviously different from other rational frequencies. One way to see the special character of $1/3$ resonance is by analysis of the b coefficients in the Lindstedt series, but we postpone that to the next section. This has a profound effect on the transformation properties of $\rho_H(\nu)$.

4. Transformation properties of $\rho_S(\nu)$ and $\rho_H(\nu)$

The radi $\rho_S(\nu)$ and $\rho_H(\nu)$ as functions of ν are very complicated fractals, with the structure apparently similar to the corresponding functions for RIT. Therefore, we expect that the singularities of $\rho_S(\nu)$ (or $\rho_H(\nu)$) at different rationals m/n have the same form. The functions are invariant under $\nu \to \nu + 1$, and $\nu \to -\nu$, so we have to investigate only the behaviour under $\nu \to \nu^{-1}$. As we shall see, $\rho_H(\nu)$ has some special features, which are a consequence of $m/n = 1/3$ being a special resonance for this map.

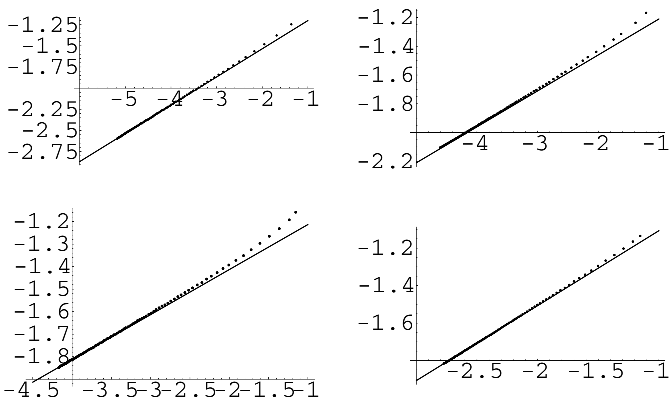

Figure 3. The graphs $(\rho_S(\nu^{-1}, \rho_S(\nu))$ (dots) and $(x, mx/n + b(m/n))$ (line) for the complex quadratic maps with: (*a*) $m/n = 1/3$, (*b*) $m/n = 1/4$, (*c*) $m/n = 1/5$ and (*d*) $m/n = 2/5$.

Let us consider first $\rho_S(\nu)$. If the structure of the leading singularity near m/n is the same for all m/n we can write

$$\begin{aligned} &\ln \rho_s(\nu) \approx f_1(m/n) \ln \rho_S(\nu^{-1}) + f_2(m/n) \\ &\nu \to \frac{m}{n} \qquad \nu^{-1} \to \frac{n}{m} \end{aligned} \tag{39}$$

where $f_1(m/n)$ and $f_2(m/n)$ are bounded functions of m/n. The notation $\approx$ is to be understood in the sense that the ratio of the left- and the right-hand sides tends to unity when $\nu \to \frac{m}{n}$ and consequently $\nu^{-1} \to \frac{n}{m}$.

In order to check relation (39) one can calculate $\rho_S(\nu_i)$ and $\rho_S\left(\nu_i^{-1}\right)$ for a sequence of noble numbers approaching a rational m/n. Plotting $\rho_S(\nu_i)$ versus $\rho_S\left(\nu_i^{-1}\right)$ along such a sequence will show if relation (39) is indeed linear. Furthermore, if (39) is true, one can obtain $f_1(m/n)$ by estimating the slope of the linear fit through the points $\left(\rho_S\left(\nu_i^{-1}\right), \rho_S(\nu_i)\right)$. Figures 3(*a*)–(*d*) illustrate a sample of such calculations for few m/n.

The graph $\left(\rho_S(\nu_i), \rho_S\left(\nu_i^{-1}\right)\right)$ is always linear, and, furthermore, $f_1(m/n) = m/n$, for all m/n that we have tested (much more than is shown in the figures). Thus, we conclude that $\ln \rho_S(\nu)$ and $\nu \ln \rho_S(\nu^{-1})$ could cancel each other's leading singularity to produce a bounded function of ν. In complete analogy with the case of RIT we define

$$L_{1S}(\nu) = \ln \rho_S(\nu) - \nu \ln \rho_S(\nu^{-1}). \tag{40}$$

Numerically calculated $L_1(\nu)$ for 5000 noble numbers of the form $\nu_j = j\gamma - [j\gamma]$ is shown in figure 4, together with $L_{0S}(\nu) = \ln \rho_S(\nu)$. The meaning of the circles will be explained later. We show the function only on (0, 1). The figure indicates that $L_1(\nu)$ is bounded at all rationals in (0, 1) and that the limit when $\nu \to 0$ exists and is equal to $-\infty$. Furthermore, $L_1(\nu)$ looks continuous, in complete analogy with the corresponding function for the case of RIT. However, cusp-like, finite, singularities due to the infinite first derivative at $\nu = m/n$ are different from the analysed cases of RIT.

Let us now consider $\rho_H(\nu)$ of the special solutions for the quadratic real maps. Following the same argument we analysed graphs $\left(\rho_H(\nu_i), \rho_H\left(\nu_i^{-1}\right)\right)$ for sequences of nobles ν_i approaching various rationals. The graphs are presented in figures 5(*a*)–(*d*) together with the corresponding linear fits by the lines $y = xm/n + b$.

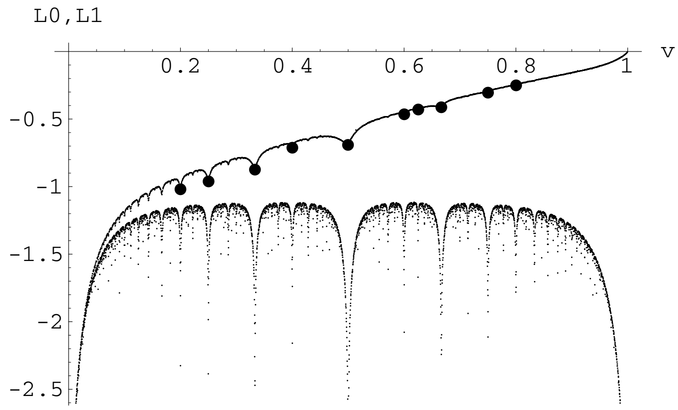

Figure 4. The functions $L_{1S}(\nu)$ (upper curve), $L_{0S}(\nu)$ (lower curve) and $L_{1S}(m/n)$ (circles) calculated with formula (45).

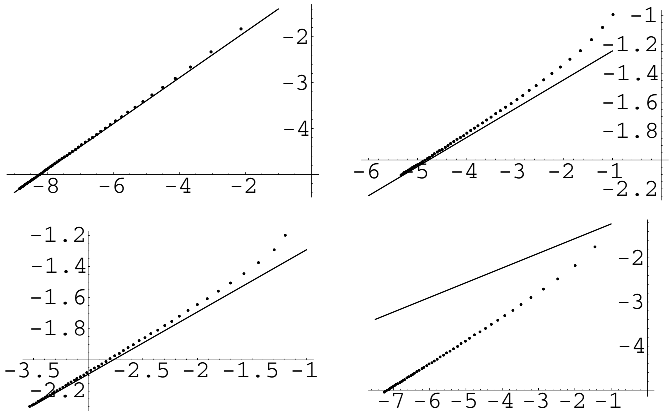

Figure 5. The graphs $(\rho_H(\nu^{-1}, \rho_H(\nu))$ (dots) and $(x, mx/n + b(m/n))$ (line) for the special solutions of the real area-preserving quadratic maps: (*a*) $m/n = 1/2$, (*b*) $m/n = 1/5$, (*c*) $m/n = 2/5$ and (*d*) $m/n = 1/3$.

By inspecting these figures (and many more that have been calculated) we conclude that for all (m, n) relatively prime and $m \neq 3 \neq n$ we have the relation

$$\ln \rho_H(\nu) \approx \frac{m}{n} \ln \rho_H(\nu^{-1}) + A(m/n) \qquad \nu \to m/n \text{ as specified.} \tag{41}$$

On the other hand, if either m or n is equal to 3 we still have a linear relation but $f_1(m/n) \neq m/n$.

Proceeding, on the same lines as before, we defined

$$L_{1H}(\nu) = \ln \rho_H(\nu) - \nu \ln \rho_H(\nu^{-1}). \tag{42}$$

The function is illustrated in figures 6(*a*) and (*b*), together with $L_{0H}(\nu) = \ln \rho_H(\nu)$. The circles will be explained later.

First, we observe that $L_1(\nu)$ is bounded at all rationals m/n such that neither m nor n is equal to 3. On the other hand, it obviously diverges as ν approaches $1/3, 2/3$ and $3/4, 3/5, 3/7, \ldots$. Actually $L_1(\nu)$ looks continuous on all intervals of the form $(3/(n+1), 3/n), n = 3, 4, \ldots$. The function has finite singularities, similar to those of

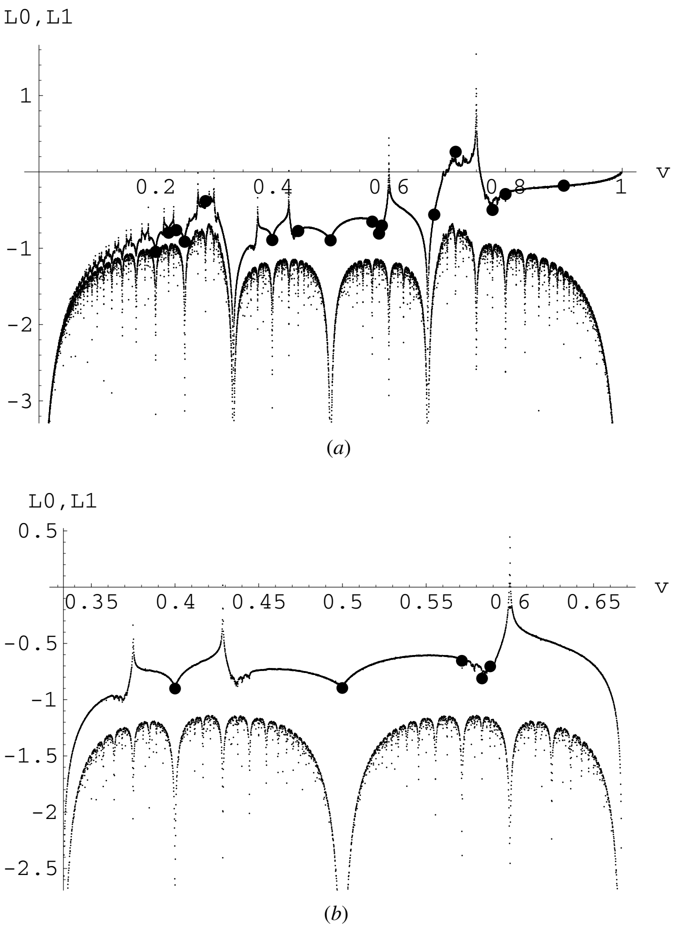

Figure 6. (*a*) The functions $L_{1H}(\nu)$ (upper curve), $L_{0H}(\nu)$ (lower curve) and $L_{1H}(m/n)$ (circles) calculated with formula (46). (*b*) The same as in (*a*) but on the interval $(1/3, 2/3)$.

$L_{1S}(\nu)$, at all m/n inside each of the intervals, and also for $m/n \in (1/3, 2/3)$. On the boundaries of these intervals the function diverges to infinity.

The behaviour of $L_{1H}(\nu)$ at the points with infinite singularity is related to the well-known special status of the 1/3 resonance of the quadratic map, which is carried over into $L_{1H}(\nu)$ via either $\ln \rho_H(\nu)$ or via $\ln \rho_H(\nu^{-1})$.

The functions $L_{1S/H}(\nu)$, introduced by (40) and (42), give relations between the most important terms in the Lindstedt series of VIT with frequencies ν and ν^{-1}. All b-coefficients of such tori have the form of sums of products of inverse powers of the divisors $D(n, \nu)$, given by (28) or (34). From the form of the divisors we see that when $\nu \to m/n$ the dominant coefficients for the Siegel map are $b_{n+1}(\nu)$, and for the real quadratic map the most important coefficients are $b_{n+1}(\nu)$ and $b_{n-1}(\nu)$. In the limit $\nu \to m/n$ precisely these b coefficients diverge, but the ratio

$$\frac{\left|b_{m+1}(\nu^{-1})\right|}{|b_{n+1}(\nu)|} \tag{43}$$

or

$$\frac{\left|b_{m+1}(\nu^{-1})\right| + \left|b_{m-1}(\nu^{-1})\right|}{|b_{n+1}(\nu)| + |b_{n-1}(\nu)|} \tag{44}$$

for map (1) or (2), respectively, is finite in this limit, as can be easily checked by explicit computation of the *b*-coefficients. Expressions (43) and (44) are related to the corresponding L_1 functions by the following formulae:

$$\lim_{\nu\to m/n} L_{1S}(\nu) \equiv L_{1S}(m/n) = \lim_{\nu\to m/n} \frac{1}{n} \ln\left[\frac{m}{n}\frac{\left|b_{m+1}(\nu^{-1})\right|}{|b_{n+1}(\nu)|}\right] \tag{45}$$

and

$$\lim_{\nu\to m/n} L_{1H}(\nu) \equiv L_{1H}(m/n) = \lim_{\nu\to m/n} \frac{1}{n} \ln\left[\frac{m}{n}\frac{\left|b_{m+1}(\nu^{-1})\right| + \left|b_{m-1}(\nu^{-1})\right|}{|b_{n+1}(\nu)| + |b_{n-1}(\nu)|}\right] \tag{46}$$

which are analogous to the corresponding formulae for RIT (see (22) and [5, 6]).

Formulae (45) and (46) are illustrated in figure 4 for (45) and figures 6(*a*) and (*b*) for (46), where circles denote $L_{1S/H}(m/n)$ calculated by formula (45) or (46), and dots are values of $L_{1S/H}(m/n)$ calculated from definition (27) or (37) and by direct estimates of $\rho_{S/H}(\nu)$ for noble ν. The agreement is obvious. We believe that (45) and (46) represent exact relations.

Thus, formulae (45) and (46) enable one to calculate the functions L_{1S} and L_{1H} respectively, at few rationals m/n with small n using the perturbation theory of a finite small order.

Obviously, more than one term is needed for a good approximation of a VIT with frequency ν further away from rationals. One should expect to find simple relations between the terms corresponding to the best rational approximants of the frequencies ν and ν^{-1}, analogous to relations (45) and (46), giving the L functions of higher orders, but we do not pursue this further.

5. Conclusions

We have analysed transformation properties of the perturbation expansions with fixed frequency of vibrational invariant tori in the two simplest but nontrivial examples. The conclusions are analogous to those obtained previously for the rotational tori. The transformation properties are described by the corresponding functions $L_{1S}(\nu)$ or $L_{1H}(\nu)$. The first one is bounded except at 0 (and infinity) and is continuous, with cusp-like finite singularities at each rational. $L_{1H}(\nu)$ has infinite singularities at all rationals m/n such that either m or n is equal to 3 or 0. However, it is bounded and continuous inside the intervals bounded by the infinite singularities, for example, on all intervals $(3/(n+1), 3/n)$, $n \geqslant 3$, and on $(1/3, 2/3)$. $L_{1H}(\nu)$ also has cusp-like finite singularities at all finite rationals m/n such that $m \neq 3 \neq n$.

All these results are a consequence of the same form of the leading singularities at different rationals (except 1/3 for the real quadratic map) in the corresponding coefficients of the perturbation expansion. This property of the fixed frequency perturbation series enables one to compare the most important terms in the expansion of a vibrational tori with the frequency near a rational by just properly rescaling the perturbation parameter k. For tori with frequencies far from rationals one would have to look for relations between more than two coefficients, probably as in the case of RIT [6], but we have not done that. Let us mention that the rescaling of the parameter k in the examples of vibrational tori treated here amounts to changing the distance from the elliptic fixed point at the origin.

Further research is required to fully understand the transformation properties of the perturbation expansions for the vibrational tori, but our results indicate that there are simple and useful relations, similar to the case of the rotational motion.

Acknowledgments

I would like to thank Professor T Bountis and Professor A Rauh for their warm hospitality and very useful discussions. Thanks are also due to DAAD (Deutsher Akademischer Austauchdienst) and to the Mathematics Department of the University of Patras, Greece, for their generous support.

References

[1] Arnold V I, Kozlov V V and Neistadt A I 1988 *Mathematical Aspects of Classical and Celestial Mechanics* (Berlin: Springer)
[2] Meiss J D 1992 Symplectic maps, variational principles and transport *Rev. Mod. Phys.* **64** 795
[3] MacKay R S 1985 *Transition to Chaos for Area-preserving Maps (Springer Lecture Notes in Physics vol 247)* (Berlin: Springer)
[4] Buric N, Percival I C and Vivaldi F 1990 Critical functions and modular smoothing *Nonlinearity* **3** 21
[5] Buric N and Percival I C 1991 Modular smoothing and finite perturbation theory *Nonlinearity* **4** 981
[6] Buric N and Percival I C 1994 Modular smoothing and KAM tori *Physica* D **71**
[7] Buric N and Mudrinic M 1998 Modular smoothing of action *J. Phys. A: Math. Gen.* **31** 1875
[8] Gentile G and Mastropietro V 1996 Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics: a review with some applications *Rev. Math. Phys.* **8** 393
[9] Percival I C 1979 Variational principles for invariant tori and cantori *Nonlinear Dynamics and the Beam–Beam Interaction* ed M Month and J C Herrera (*Am. Inst. Phys. Conf. Proc.* **57**) (New York: AIP) p 302
[10] Mather J N 1982 Existance of quasi-periodic orbits of twist homeomorphisms of the annulus *Topology* **21** 457
[11] Aubry S 1983 The twist maps, the extended Frankel–Kantorova model and the Devil's staircase *Physica* D **7** 240
[12] Greene J M and Percival I C 1981 Hamiltonian maps in the complex plane *Physica* D **3** 530
[13] Percival I C 1982 Chaotic boundary of a Hamiltonian map *Physica* D **6** 67
[14] Escande D F 1985 Stochastisity in classical Hamiltonian systems *Phy. Rep.* **121** 126
[15] Celletti A and Cherchia L 1988 Construction of analytic KAM surfaces and effective stability bounds *Commun. Math. Phys.* **118** 119
[16] Chandre C, Govin M and Jauslin H R 1996 Kolmogorov–Arnold–Moser renormalization-group approach to the breakup of invariant tori in Hamiltonian systems *Phys. Rev.* E **67** 6612
[17] Berretti A and Chierchia L 1990 On the complex analytic structure of the golden invariant curve for the standard map *Nonlinearity* **3** 39
Berretti A, Celletti A, Chiercia L and Falcolini C 1992 Natural boundaries for area-preserving twist maps *J. Stat. Phys.* **66** 1613
[18] Falcolini C and de la Llave R 1992 Numerical calculations of domains of analyticity for perturbation theories with the presence of small divisors *J. Stat. Phys.* **67**
[19] Davie A M 1994 Critical function for the semi-standard map *Nonlinearity* **7** 219
[20] Berretti A and Gentile G 1999 Scalin properties for the radius of convergence of a Lindstedt series: the standard map *J. Math. Pure Appl.* **78** 159
[21] Berretti A and Marmi S 1994 Scaling near resonances and complex rotation numbers for the standard map *Nonlinearity* **7** 603
[22] Berretti A and Gentile G 1998 Bruno function and the standard map *Preprint* chao-dyn/9810035
[23] Marmi S and Stark J 1992 On the standard map critical function *Nonlinearity* **5** 743
[24] Buric N, Mudrinic M and Timotijevic D 1996 Efficient and accurate calculations of the stability bounds in Hamiltonian systems *Phys. Rev.* E **54** 1463
[25] Rüssmann H 1967 Über die normalform analytisher hamiltonsher differentialgleichungen in der nahe einer gleichgewichtslöung *Math. Ann.* **169** 55
[26] Marmi S 1990 Critical functions for complex analytic maps *J. Phys. A: Math. Gen.* **23** 3447

[27] Marmi S 2000 An introduction to small divisors problems *Preprint* mp-arc/a/00-381
[28] Yoccoz J-C 1995 Theorème de Siegel, nombres de Brjuno and pôlinomes quadratiques *Astérisque* **231** 3
[29] Percival I C 1984 The transition to chaos for a special solution of the area-preserving quadratic map *Physica* D **14** 49
[30] Giovannozzi M and Marmi S 1989 Existence of complex invariant circles in the quadratic area-preserving map *Rend. Mat.* **9** 457
[31] Meiss D J 2000 Generic twistless bifurcations *Nonlinearity* **13** 203